

FRACTIONAL DIFFERENTIATION AND ITS APPLICATIONS

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Propagation Speed of the Maximum of the Fundamental Solution to the Fractional Diffusion-Wave Equation

Yuri LUCHKO⁽¹⁾, Francesco MAINARDI⁽²⁾ and Yuriy POVSTENKO⁽³⁾

¹Department of Mathematics,

Beuth Technical University of Applied Sciences, Berlin, 13353 Germany

E-mail: luchko@beuth-hochschule.de

⁽²⁾ Department of Physics, University of Bologna, and INFN

Via Irnerio 46, I-40126 Bologna, Italy

E-mail: francesco.mainardi@unibo.it; francesco.mainardi@bo.infn.it

⁽³⁾ Institute of Mathematics and Computer Science,

Jan Dlugosz University in Czestochowa, Czestochowa, 42-200 Poland

E-mail: j.povstenko@ajd.czyst.pl

Abstract

In this paper, the one-dimensional time-fractional diffusion-wave equation with the fractional derivative of order α , $1 < \alpha < 2$ is revisited. This equation interpolates between the diffusion and the wave equations that behave quite differently regarding their response to a localized disturbance: whereas the diffusion equation describes a process, where a disturbance spreads infinitely fast, the propagation speed of the disturbance is a constant for the wave equation. For the time-fractional diffusion-wave equation, the propagation speed of a disturbance is infinite, but its fundamental solution possesses a maximum that disperses with a finite speed. In this paper, the fundamental solution of the Cauchy problem for the time-fractional diffusion-wave equation, its maximum location, maximum value, and other important characteristics are investigated in detail. To illustrate analytical formulas, results of numerical calculations and plots are presented. Numerical algorithms and programs used to produce plots are discussed.

Key Words and Phrases: Time-fractional diffusion-wave equation, Cauchy problem, fundamental solution, Wright function, Mainardi function, propagation speed, maximum of fundamental solution, numerical calculation of the Green function

1 Introduction

Evolution equations related to phenomena intermediate between diffusion and wave propagation have attracted attention of a number of researchers since 1980's. This kind of phenomena is known to occur in viscoelastic media that combine characteristics of solid-like materials that exhibit waves propagation and fluid-like materials that support diffusion processes. In particular, analysis and results presented in [Pipkin (1986)] and [Kreis and Pipkin (1986)] should be mentioned. Being unaware of an interpretation of evolution equations by means of fractional calculus, these authors still could provide an interesting example of relevance of the intermediate phenomena for models in continuum mechanics.

Nowadays it is well recognized that evolution equations can be interpreted as differential equations of fractional order at the time when some hereditary mechanisms of power-law type are present in diffusion or wave phenomena. This has been shown in [Mainardi and Tomirotti (1997)] and more recently in [Mainardi (2010)], where propagation of pulses in linear viscoelastic media governed by constitutive equations of fractional order has been revisited.

For analysis of the evolution equations of the type mentioned above, methods and tools of fractional calculus, integral transforms, and higher transcendental functions have been employed in the pioneering papers [Wyss (1986)], [Schneider and Wyss (1989)], [Fujita (1990)], [Gorenflo and Rutman (1994)], [Kochubei (1989), Kochubei (1990)], and in the book [Prüss (1993)]. We also mention the papers [Mainardi (1994), Mainardi (1996a), Mainardi (1996b)] and [Mainardi and Tomirotti (1995)], where fundamental solutions of the evolution equations related to phenomena intermediate between diffusion and wave propagation have been expressed in terms of some auxiliary functions of the Wright type that sometimes are referred to as Mainardi functions, see i.e. [Podlubny (1999)], [Gorenflo et al. (1999), Gorenflo et al. (2000)]. These functions as well as some techniques and methods of fractional calculus, integral transforms, and higher transcendental functions will be used in our analysis.

It is well known that diffusion and wave equations behave quite differently regarding their response to a localized disturbance: whereas the diffusion equation describes a process, where a disturbance spreads infinitely fast, the propagation speed of the disturbance is constant for the wave equation. In a certain sense, the time-fractional diffusion-wave equation interpolates between these two different responses. On the one hand, the support of the solution to this equation is not compact on the real line for each $t > 0$ for a non-negative disturbance that is not identically equal to zero, i.e. its response to a localized disturbance spreads infinitely fast (see [Fujita (1990)]). On the other hand, the fundamental solution of the time-fractional diffusion-wave equation possesses a maximum that disperses with a finite speed similar to the behavior of the fundamental solution of the wave equation. The problem to describe location of maximum of the fundamental solution of the Cauchy problem for the one-dimensional time-fractional diffusion-wave equation of order α , $1 < \alpha < 2$ was considered for the first time in [Fujita (1990)]. Fujita proved that the fundamental solution takes its maximum at the point $x_* = \pm c_\alpha t^{\alpha/2}$ for each $t > 0$, where $c_\alpha > 0$ is a constant determined by α . Recently, another proof of this formula for the maximum location along with numerical results for the constant c_α for $1 < \alpha < 2$ were presented in [Povstenko (2008)]. In this paper, we provide an extension and consolidation of these results along with some new analytical formulas, numerical algorithms, and pictures.

The rest of the paper is organized as follows: in the 2nd section, problem formulation and some analytical results are given. Here we revisit the results of Fujita and Povstenko and give some new insights into the problem. Especially the role of the symmetry group of scaling transformations of the time-fractional diffusion-wave equation in the maximum propagation problem is emphasized. We derive a new formula for the maximum value of the Green function for the Cauchy problem for the time-fractional diffusion-wave equation. A new characteristic of the time-fractional diffusion-wave equation - the product of the maximum location of its fundamental solution and its maximum value - is introduced. For a fixed value of α , $1 \leq \alpha \leq 2$, this product is a constant for all $t > 0$ that depends only on α . The product is equal to zero for the diffusion equation and to infinity for the wave equation, whereas it is finite, positive, and laying between these extreme values for the time-fractional diffusion-wave equation that justifies the fact that the time-fractional diffusion-wave equation interpolates between the diffusion and the wave equations. The 3rd section is devoted to presentation of the numeri-

cal algorithms used to calculate the fundamental solution and its important characteristics including the location of its maximum, its propagation speed, and the maximum value. Results of numerical calculations and plots are presented and discussed in detail.

2 Problem formulation and analytical results

This section is devoted to the problem formulation and some important analytical results. In particular, several representations of the fundamental solution of the Cauchy problem for the time-fractional diffusion-wave equation in form of series and integrals are given. These representations are used to derive explicit formulas for the maximum location, maximum value, and the propagation speed of the maximum point. Besides, we give a new proof of the fact that a response of the time-fractional diffusion-wave equation to a localized disturbance spreads infinitely fast like in the case of the diffusion equation.

2.1 Problem formulation

In this paper, we deal with the family of evolution equations obtained from the standard diffusion equation (or the D'Alembert wave equation) by replacing the first-order (or the second-order) time derivative by a fractional derivative of order α with $1 \leq \alpha \leq 2$, namely

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2}, \quad (1)$$

where $x \in \mathbb{R}$, $t \in \mathbb{R}^+$ denote the space and time variables, respectively.

In (1), $u = u(x, t)$ represents the response field variable and the fractional derivative of order α , $n - 1 < \alpha < n$, $n \in \mathbb{N}$ is defined in the Caputo sense:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - \tau)^{n - \alpha - 1} \frac{\partial^n u(\tau)}{\partial \tau^n} d\tau, \quad (2)$$

where Γ denotes the Gamma function. For $\alpha = n$, $n \in \mathbb{N}$, the Caputo fractional derivatives is defined as the standard derivative of order n .

In order to guarantee existence and uniqueness of a solution, we must add to (1) some initial and boundary conditions. Denoting by $f(x)$, $x \in \mathbb{R}$ and

$h(t)$, $t \in \mathbb{R}^+$ sufficiently well-behaved functions, the Cauchy problem for the time-fractional diffusion-wave equation with $1 \leq \alpha \leq 2$ is formulated as follows:

$$\begin{cases} u(x, 0) = f(x), & -\infty < x < +\infty; \\ u(\mp\infty, t) = 0, & t > 0. \end{cases} \quad (3)$$

If $1 < \alpha \leq 2$, we must add to (3) the initial value of the first time derivative of the field variable, $u_t(x, 0)$, since in this case the Caputo fractional derivative is expressed in terms of the second order time derivative. To ensure continuous dependence of the solution with respect to the parameter α we agree to assume

$$u_t(x, 0) = 0, \quad \text{for } 1 < \alpha \leq 2.$$

In view of our subsequent analysis we find it convenient to set $\nu := \alpha/2$, so that $1/2 \leq \nu \leq 1$ for $1 \leq \alpha \leq 2$.

For the Cauchy problem, we introduce the so-called Green function $\mathcal{G}_c(x, t; \nu)$, which represents the respective fundamental solution, obtained when $f(x) = \delta(x)$, δ being the Dirac δ -function. As a consequence, the solution of the Cauchy problem is obtained by a space convolution according to

$$u(x, t; \nu) = \int_{-\infty}^{+\infty} \mathcal{G}_c(x - \xi, t; \nu) f(\xi) d\xi.$$

It should be noted that $\mathcal{G}_c(x, t; \nu) = \mathcal{G}_c(|x|, t; \nu)$ since the Green function of the Cauchy problem turns out to be an even function of x . This means that we can restrict our investigation of the function \mathcal{G}_c to non-negative values $x \geq 0$.

For the standard diffusion equation ($\nu = 1/2$) it is well known that

$$\mathcal{G}_c(x, t; 1/2) := \mathcal{G}_c^d(x, t) = \frac{t^{-1/2}}{2\sqrt{\pi}} e^{-x^2/(4t)}.$$

In the limiting case $\nu = 1$ we recover the standard wave equation, for which we get

$$\mathcal{G}_c(x, t; 1) := \mathcal{G}_c^w(x, t) = \frac{1}{2} [\delta(x - t) + \delta(x + t)].$$

In the case $1/2 < \nu < 1$, the Green function \mathcal{G}_c will be determined in the next subsection by using the technique of the Laplace and the Fourier transforms. The representations of the Green function \mathcal{G}_c are of course not new and have

been discussed in [Mainardi (1994)]-[Mainardi (2011)] to mention only a few of many papers devoted to this topic.

In this paper, we are interested in investigation of some important characteristics of the Green function \mathcal{G}_c including location of its maximum point, its propagation speed, and its maximum value.

2.2 Representations of the Green function

Following [Mainardi (1994)]-[Mainardi (2011)], some representations of the Green function \mathcal{G}_c in form of integrals and series are presented and discussed in this subsection.

In [Mainardi (1994)], the Laplace and Fourier transforms technique was employed to deduce the following representation for the Green function \mathcal{G}_c for $x > 0$ and $\frac{1}{2} < \nu < 1$:

$$2\nu x \mathcal{G}_c(x, t; \nu) = F_\nu(r) = \nu r M_\nu(r), \quad (4)$$

where

$$r = x/t^\nu > 0$$

is the similarity variable and

$$F_\nu(r) := \frac{1}{2\pi i} \int_{Ha} e^{\sigma - r\sigma^\nu} d\sigma, \quad M_\nu(r) := \frac{1}{2\pi i} \int_{Ha} \frac{e^{\sigma - r\sigma^\nu}}{\sigma^{1-\nu}} d\sigma$$

are the two auxiliary functions nowadays referred to in the literature of Fractional Calculus as the Mainardi functions, and Ha denotes the Hankel path properly defined for the representation of the inverse of the Gamma function.

Let us note that the similarity variable $r = x/t^\nu$ plays a very important role in our analysis of location of a maximum point of the Green function \mathcal{G}_c . In its turn, the form of the similarity variable can be explained by the Lie group analysis of the time-fractional diffusion-wave equation (1). As shown in [Buckwar and Luchko (1998)], [Luchko and Gorenflo (1998)] and [Gorenflo et al. (2000)], symmetry groups of scaling transformations for the time- and space-fractional partial differential equations have been constructed. In particular, it has been proved in [Buckwar and Luchko (1998)] that the only invariant of the symmetry group T_λ of scaling transformations of the

time-fractional diffusion-wave equation (1) has the form $\eta(x, t, u) = x/t^\nu$ that explains the form of the scaling variable.

Using the well known representation of the Wright function, which reads (in our notation) for $z \in \mathbb{C}$

$$W_{\lambda, \mu}(z) := \frac{1}{2\pi i} \int_{Ha} \frac{e^{\sigma + z\sigma^{-\lambda}}}{\sigma^\mu} d\sigma = \sum_{n=0}^{\infty} \frac{z^n}{n! \Gamma(\lambda n + \mu)}, \quad (5)$$

where $\lambda > -1$ and $\mu > 0$, we recognize that the auxiliary functions are related to the Wright function according to

$$F_\nu(z) = W_{-\nu, 0}(-z) = \nu z M_\nu(z), \quad M_\nu(z) = W_{-\nu, 1-\nu}(-z). \quad (6)$$

The formula (6) along with (5) provides us with the series representations of the Mainardi functions and thus of the Green function \mathcal{G}_c (for $x > 0$):

$$\mathcal{G}_c(x, t; \nu) = \frac{1}{2^\nu x} F_\nu(r) = \frac{1}{2 t^\nu} M_\nu(r) = \frac{1}{2 t^\nu} \sum_{n=0}^{\infty} \frac{(-x/t^\nu)^n}{n! \Gamma(-\nu n + 1 - \nu)}. \quad (7)$$

The formulas (6)-(7) can be used to give a new proof of the known fact that the support of the Green function \mathcal{G}_c is not compact on the real line for each $t > 0$, i.e. that a response of the time-fractional diffusion-wave equation with $1/2 < \nu < 1$ to a localized disturbance spreads infinitely fast. Indeed, because the Wright function (5) is an analytical function for $\lambda > -1$ and $\mu > 0$ (see e.g. [Gorenflo et al. (1999)]) that is not identically equal to zero ($W_{\lambda, \mu}(0) = 1/\Gamma(\mu) > 0$), the set of its zeros is discrete and has no finite limit points in the complex plane and thus on the real line. This means that the support of the function $\mathcal{G}_c(x, t; \nu) = W_{-\nu, 1-\nu}(-x/t^\nu)$ is not compact on the real line for each $t > 0$. This fact was proved in [Fujita (1990)] using the representation (7) of \mathcal{G}_c as a function depending on the similarity variable and the asymptotics of this function.

Finally we mention another integral representation of the Green function \mathcal{G}_c that can be found e.g. in [Mainardi et al. (2001)] or [Povstenko (2008)]:

$$\mathcal{G}_c(x, t; \nu) = \frac{1}{\pi} \int_0^\infty E_{2\nu}(-\kappa^2 t^{2\nu}) \cos(x\kappa) d\kappa, \quad (8)$$

where $E_\alpha(z)$ is the Mittag-Leffler function defined by the series

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad \alpha > 0. \quad (9)$$

The representation (8) can be easily obtained by transforming the Cauchy problem for the equation (1) into the Laplace-Fourier domain using the known formula

$$\mathcal{L} \left\{ \frac{d^\alpha u(t)}{dt^\alpha}; s \right\} = s^\alpha \mathcal{L} \{u(t); s\} - \sum_{k=0}^{n-1} u^{(k)}(0^+) s^{\alpha-1-k}, \quad n-1 < \alpha \leq n, \quad (10)$$

with $n \in \mathbb{N}$, for the Laplace transform of the Caputo fractional derivative. This formula together with the standard formulas for the Fourier transform of the second derivative and of the Dirac δ -function lead to the representation

$$\widehat{\mathcal{G}}_c(\kappa, s, \nu) = \frac{s^{2\nu-1}}{s^{2\nu} + \kappa^2}, \quad \nu = \alpha/2 \quad (11)$$

of the Laplace-Fourier transform $\widehat{\mathcal{G}}_c$ of the Green function \mathcal{G}_c . Using the well-known Laplace transform formula (see e.g. [Podlubny (1999)])

$$L \{E_\alpha(-t^\alpha); s\} = \frac{s^{\alpha-1}}{s^\alpha + 1}$$

and applying to the R.H.S of the formula (11) first the inverse Laplace transform and then the inverse Fourier transform we obtain the integral representation (8) if we take into consideration the fact that the Green function of the Cauchy problem is an even function of x that follows from the formula (11).

2.3 Maximum points of the Green function \mathcal{G}_c

In Fig. 1, several plots of the Green function $\mathcal{G}_c(x; \nu) := \mathcal{G}_c(x, 1; \nu)$ for different values of the parameter ν ($\nu = \alpha/2$) are presented (see the next section for description of numerical algorithms and programs used to calculate the numerical values of the Green function). It can be seen that each Green function has an only maximum and that location of the maximum point changes with the value of ν . In Fig. 2, the Green function $\mathcal{G}_c(x, t; \nu)$ is plotted for $\nu = 0.875$ from different perspectives. The plots show that both the location of maximum and the maximum value depend on the time $t > 0$: whereas the maximum value decreases with the time (Fig. 2, right), the x -coordinate of the maximum location becomes even larger (Fig. 2, left). Below we give a surprisingly easy explicit formula that connects the location of maximum and the maximum value for all times $t > 0$ for a fixed ν , $1/2 \leq \nu \leq 1$.

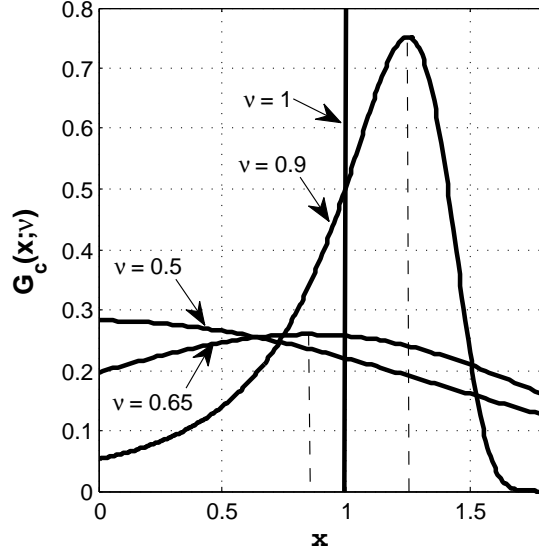


Figure 1: Green function $\mathcal{G}_c(x; \nu) := \mathcal{G}_c(x, 1; \nu)$: Plots for several different values of ν

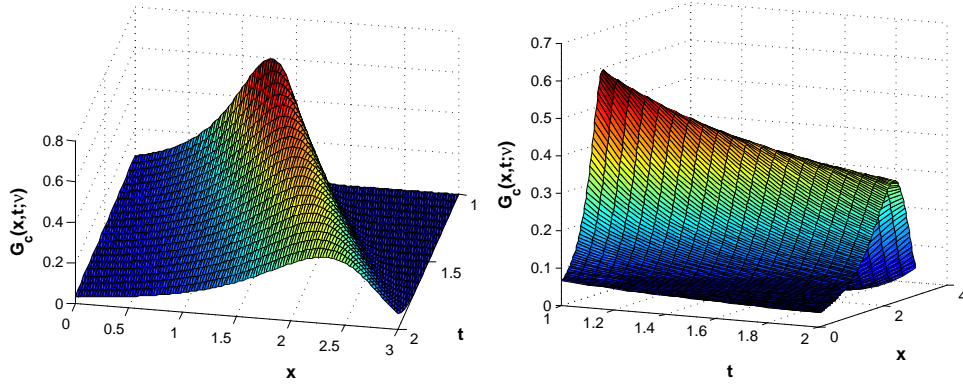


Figure 2: Green function $\mathcal{G}_c(x, t; \nu)$: Plots for $\nu = 0.875$ from different perspectives

The aim of this subsection is to present some analytical formulas that describe both the location of the maximum of the Green function \mathcal{G}_c , its value, and their interconnection, as well as the propagation speed of the maximum location.

In the paper [Fujita (1990)], an elegant proof of the fact that the Green function $\mathcal{G}_c(x, t; \nu)$ of the Cauchy problem takes its maximum at the point $x_*(t, \nu) = \pm c_\nu t^\nu$ for each $t > 0$, where $c_\nu > 0$ is a constant determined by ν , $1/2 < \nu < 1$, has been presented. His reasoning was as follows: Let us consider the Green function at the point $t = 1$ and for $x \geq 0$: $\mathcal{G}_c(x; \nu) := \mathcal{G}_c(x, 1; \nu)$. For $1/2 < \nu < 1$ the function $\mathcal{G}_c(x; \nu)$ is a stable probability density and the stable densities are all unimodal (see [Lukacs (1960)]). This means that $\mathcal{G}_c(x; \nu)$ takes its maximum at a certain point $x_{**} = c_\nu$ with a constant c_ν depending on ν . It follows from the formula (4) that

$$\mathcal{G}_c(x; \nu) = \frac{1}{2} M_\nu(x) \quad (12)$$

and

$$\mathcal{G}_c(x, t; \nu) = \frac{t^{-\nu}}{2} M_\nu(xt^{-\nu}) = t^{-\nu} \mathcal{G}_c(xt^{-\nu}; \nu). \quad (13)$$

Because the function $\mathcal{G}_c(x; \nu)$ takes its maximum at the point $x_{**} = c_\nu$, the function $\mathcal{G}_c(x, t; \nu)$ has to take its maximum at the point x_* that satisfies the relation $x_* t^{-\nu} = x_{**} = c_\nu$ due to the formula (13). Thus the maximum point of the Green function $\mathcal{G}_c(x, t; \nu)$ is moving with the time according to the formula

$$x_*(t) = c_\nu t^\nu, \quad \nu = \alpha/2. \quad (14)$$

As we see, the main argument in the Fujita's proof is dependence of the Green function from the similarity variable $xt^{-\nu}$.

This argument can be used to give an analytical proof of the relation (14) following an idea presented in [Povstenko (2008)]. It should be noted that derivation of (14) given in [Povstenko (2008)] is based on the integral representation (8) and contains some divergent integrals that should be interpreted in one or another generalized sense. To avoid this, we present here another proof of (14) based on the representation (7) of the the Green function via the Mainardi function and not on the integral representation (8).

Because the Mainardi function M_ν is an analytical function for $\nu < 1$ as a particular case of the Wright function and because of (7), there exist partial derivatives of the Green function $\mathcal{G}_c(x, t; \nu)$ of arbitrary orders for $t > 0$, $x > 0$

and we can use the standard analytical method for finding its extremum points. We first fix a value $t > 0$ and look for the critical points of the Green function $\mathcal{G}_c(x, t; \nu)$ that are determined as solutions to the equation

$$\frac{\partial}{\partial x} \mathcal{G}_c(x, t; \nu) = \frac{t^{-2\nu}}{2} M'_\nu(xt^{-\nu}) = 0 \quad (15)$$

or to the equation

$$M'_\nu(xt^{-\nu}) = 0. \quad (16)$$

Now we try to find a function $x_* = x_*(t)$ that gives a solution to the equation (16) for each $t > 0$. The equation (16) can be interpreted as an implicit function that determines the function $x_* = x_*(t)$. Let us find the time-derivative of x_* as a derivative of an implicit function:

$$x'_*(t) = -\frac{\frac{\partial}{\partial t} M'_\nu(xt^{-\nu})}{\frac{\partial}{\partial x} M'_\nu(xt^{-\nu})} = -\frac{-\nu t^{-\nu-1} x M''_\nu(xt^{-\nu})}{t^{-\nu} M''_\nu(xt^{-\nu})} = \nu \frac{x}{t}.$$

We thus obtained a simple differential equation for $x_*(t)$ with the solution $x_*(t) = Ct^\nu$, where $C = x_*(1) = c_\nu$ that is in accordance with the formula (14). Let us mention that the same method can be applied for any twice-differentiable function that depends on the similarity variable $xt^{-\nu}$ or another one in form of product of the power functions in x and t . As it is known, the Green functions for many linear partial differential equations of fractional order possess this property and can be investigated by the method presented above.

As mentioned in [Fujita (1990)], the maximum point of the Green function $\mathcal{G}_c(x, t; \nu)$ propagates for $t > 0$ with a finite speed $v(t, \nu)$ that is determined by

$$v(t, \nu) := x'_*(t) = \nu c_\nu t^{\nu-1}. \quad (17)$$

This formula shows that for every ν , $1/2 < \nu < 1$ the propagation speed of the maximum point of the Green function \mathcal{G}_c is a decreasing function in t that varies from $+\infty$ at time $t = 0+$ to zero as $t \rightarrow +\infty$. For $\nu = 1/2$ (diffusion) the propagation speed is equal to zero because of $c_{1/2} = 0$ whereas for $\nu = 1$ (wave propagation) it remains constant and is equal to $c_1 = 1$. In Fig. 3, some plots of the propagation speed of the maximum point of the Green function \mathcal{G}_c are given for different values of ν . For large values of t , the smaller the value of ν is, the smaller is the propagation speed for the same time instant. Conversely, according to the formula (17), the smaller the value

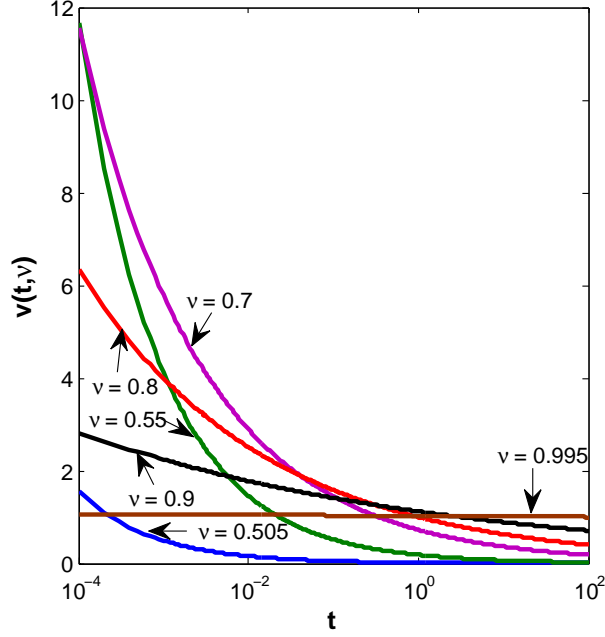


Figure 3: Propagation speed of the maximum point: Plot of $v(t, \nu)$ for different values of ν in the log-lin scale

of ν is, the bigger is the propagation speed for the same time instant when $t \rightarrow 0^+$. For example, the propagation speed for $\nu = 0.505$ becomes greater than the one for $\nu = 0.55$ for $t < 3.04E - 24$ (of course, this effect is not visible in the plot of Fig. 3).

Now we determine the maximum value of $\mathcal{G}_c(x, t; \nu)$ in dependence of time. Let us denote the maximum value by $\mathcal{G}_c^*(t; \nu)$ and find it by using the integral representation (8):

$$\mathcal{G}_c^*(t; \nu) := \mathcal{G}_c(x_*(t), t; \nu) = \frac{1}{\pi} \int_0^\infty E_{2\nu}(-\kappa^2 t^{2\nu}) \cos(c_\nu t^\nu \kappa) d\kappa. \quad (18)$$

The variables substitution $\tau = t^\nu \kappa$ reduces the integral in (18) to the form

$$\mathcal{G}_c^*(t; \nu) = \frac{t^{-\nu}}{\pi} \int_0^\infty E_{2\nu}(-\tau^2) \cos(c_\nu \tau) d\tau, \quad (19)$$

i.e. the maximum value $\mathcal{G}_c^*(t; \nu)$ of the Green function can be written in the form

$$\mathcal{G}_c^*(t; \nu) = m_\nu t^{-\nu}, \quad (20)$$

$$m_\nu = \frac{1}{\pi} \int_0^\infty E_{2\nu}(-\tau^2) \cos(c_\nu \tau) d\tau. \quad (21)$$

It follows from the relations (14) and (20) that the product

$$\mathcal{G}_c^*(t; \nu) \cdot x_*(t) = c_\nu m_\nu, \quad 0 < t < \infty \quad (22)$$

is a constant that depends only on ν or on the order α of the fractional derivative in the equation (1), i.e. that the maximum locations and the corresponding maximum values specify a certain hyperbola for a fixed value of α and for $0 < t < \infty$. This fact is rather unexpected and one should look for its explanation from the physical and/or probabilistic viewpoints. Let us note that the product $\mathcal{G}_c^*(t; \nu) \cdot x_*(t)$ is equal to zero in the case $\nu = 1/2$ (diffusion equation) because the maximum point is always located at the point $x_* = 0$ and to infinity in the case $\nu = 1$ (wave equation) because the maximum value is always equal to infinity. The product values for $1/2 < \nu < 1$ are finite and laying between these extreme values that justifies the fact that the time-fractional diffusion-wave equation interpolates between the diffusion and the wave equations.

In Fig. 4, we give some plots of the parametric curve $(x_*(t), \mathcal{G}_c^*(t; \nu))$ for $0 < t < \infty$ that is in fact a hyperbola for different values of ν . The vertex of the hyperbola tends to the point $(0, 0)$ when ν tends to $1/2$ (diffusion equation) and to infinity when $\nu \rightarrow 1$ (wave equation).

Another interesting and important curve is presented in Fig. 5, where the product $c_\nu m_\nu$ of the maximum location and the maximum value of the Green function $\mathcal{G}_c(x; \nu)$ is plotted for $1/2 < \nu < 1$. As we have seen above, the constants c_ν (maximum location of $\mathcal{G}_c(x; \nu)$) and m_ν (maximum value of $\mathcal{G}_c(x; \nu)$) are decisive for the behavior of the Green function $\mathcal{G}_c(x, t; \nu)$ for all $t > 0$ because the maximum locations and values of this function can be determined via these constants for any time point $t > 0$ (see the formulas (14) and (20)). As we can see in Fig. 5, the product $c_\nu m_\nu$ is a monotonically increasing function that takes values between 0 (diffusion equation) and $+\infty$ (wave equation). For $0.56 < \nu < 0.99$, the product varies between 0.1 and 10, i.e. it changes very slowly on this interval. For $\nu \rightarrow 1/2$ and $\nu \rightarrow 1$ the product $c_\nu m_\nu$ goes to 0 (diffusion equation) and to $+\infty$ (wave equation), respectively, very fast.

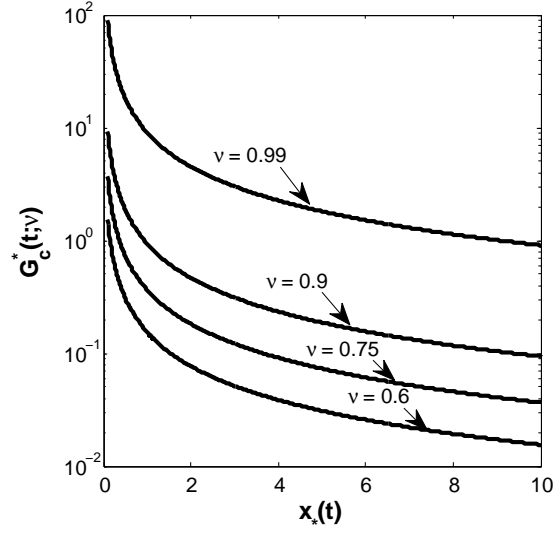


Figure 4: Maximum locations and maximum values of $\mathcal{G}_c(x, t; \nu)$ for a fixed value of ν : Plots of the parametric curve $(x_*(t), \mathcal{G}_c^*(t; \nu))$, $0 < t < \infty$ for different values of ν in the lin-log scale

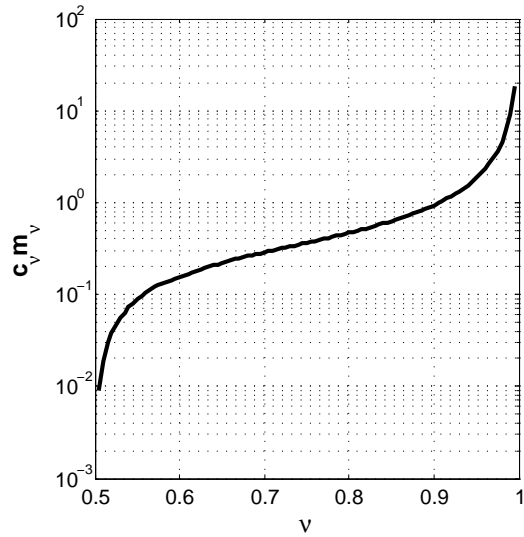


Figure 5: Product of maximum locations and maximum values of $\mathcal{G}_c(x, t; \nu)$ for a fixed time $t = 1$: Plot of $c_\nu m_\nu$, $1/2 < \nu < 1$ in the lin-log scale

3 Numerical algorithms and results

In the previous section, some analytical results regarding the location of the maximum of the Green function \mathcal{G}_c , its maximum value, and the propagation speed of the maximum point as well as the plots in Fig. 1 - Fig. 5 were presented. Because the analytical formulas derived in the previous section contain the constants c_ν and m_ν that we could not determine in analytical form, we used some numerical algorithms and MATLAB-programs for their calculation. These algorithms along with some numerical results and plots are presented in this section.

To start with, we first discuss algorithms for numerical evaluation of the Green function \mathcal{G}_c . Because \mathcal{G}_c is a particular case of the Wright function (see formula (6)), one can of course use the algorithms for the numerical evaluation of the Wright function suggested in [Luchko (2008)] to evaluate the Green function \mathcal{G}_c .

Another approach to numerical evaluation of \mathcal{G}_c we employed to produce our plots for this paper is to use the integral representation (8). To evaluate the Mittag-Leffler function E_α in (8), we applied the algorithms suggested in [Gorenflo and Luchko(2002)] and the MATLAB-programs that implement these algorithms and are available from [Matlab File Exchange (2005)]. Because the Mittag-Leffler function has for $0 < \alpha < 2$ the asymptotics (see e.g [Podlubny (1999)])

$$E_\alpha(-x) = \frac{1}{x \Gamma(1-\alpha)} + O(x^{-2}), \quad x \rightarrow +\infty, \quad (23)$$

we can estimate the length of the finite integration interval in the improper integral (8) that allows to reach the desired accuracy ϵ . Indeed, let $A \gg t^{-\nu}$ and

$$\frac{1}{\epsilon} \frac{2t^{-2\nu}}{\pi |\Gamma(1-2\nu)|} < A.$$

Then the estimate

$$|E_{2\nu}(-\kappa^2 t^{2\nu})| \leq \frac{2}{\kappa^2 t^{2\nu} |\Gamma(1-2\nu)|}$$

holds true for $\kappa > A$ because of the asymptotic expansion (23) and we have

$$\frac{1}{\pi} \left| \int_A^\infty E_{2\nu}(-\kappa^2 t^{2\nu}) \cos(x\kappa) d\kappa \right| \leq \int_A^\infty \frac{2}{\pi \kappa^2 t^{2\nu} |\Gamma(1-2\nu)|} d\kappa = \frac{1}{A} \frac{2t^{-2\nu}}{\pi |\Gamma(1-2\nu)|} < \epsilon.$$

The integral

$$\frac{1}{\pi} \int_0^A E_{2\nu}(-\kappa^2 t^{2\nu}) \cos(x\kappa) d\kappa$$

with a finite value of A can be then evaluated using any of the known quadrature formulas. If A satisfies the conditions mentioned above, we get the estimate

$$\left| \frac{1}{\pi} \int_0^A E_{2\nu}(-\kappa^2 t^{2\nu}) \cos(x\kappa) d\kappa - \mathcal{G}_c(x, t; \nu) \right| < \epsilon$$

with the desired accuracy ϵ that has been used for numerical evaluation of the Green function \mathcal{G}_c . The results of the numerical evaluation of the Green function \mathcal{G}_c for the time $t = 1$ and for different values of ν are presented in Fig. 1. In Fig. 2, 3D-plots of the Green function are given for $\nu = 0.875$, $1 \leq t \leq 2$, and $0 \leq x \leq 3$ that illustrate a typical behavior of \mathcal{G}_c .

As can be seen in Fig. 1 and as expected, $\mathcal{G}_c(x, 1; \nu)$ has a unique maximum for each $1/2 \leq \nu \leq 1$ and the maximum location changes with ν . Surprisingly, the maximum location does not always lay between zero (maximum location for the diffusion equation, $\nu = 1/2$) and one (maximum location for the wave equation, $\nu = 1$). Below we consider this phenomena in more details.

As we have seen in the previous section (formula (14)), the location of the maximum point of the Green function $\mathcal{G}_c(x, t; \nu)$ depends on the constant c_ν , i.e. on the location of its maximum for $t = 1$. It is therefore very important to calculate c_ν numerically and to visualize the dependence of c_ν on ν , $1/2 \leq \nu \leq 1$. Because we already know how to calculate the Green function $\mathcal{G}_c(x, 1; \nu)$ and because it possesses a unique maximum point $x_* = c_\nu$, it is an easy task to find the maximum location e.g. with the MATLAB Optimization Toolbox. The results of the calculations are presented in Fig. 6.

Fig. 6 shows that the curve $c_\nu = c_\nu(\nu)$ has a maximum located at the point $\nu \approx 0.85$. The value of the maximum is approximately equal to 1.28. It is interesting to note that for $0.69 \leq \nu \leq 1$ the value of c_ν is greater than or equal to one.

Finally we present results of numerical evaluation of the maximum value m_ν of the Green function $\mathcal{G}_c(x, t; \nu)$ at the time $t = 1$ as function of ν , $1/2 \leq \nu < 1$ in Fig. 6. It follows from (20) that the constant m_ν determines the maximum value of $\mathcal{G}_c(x, t; \nu)$ at any time $t > 0$. For numerical calculation of m_ν , the

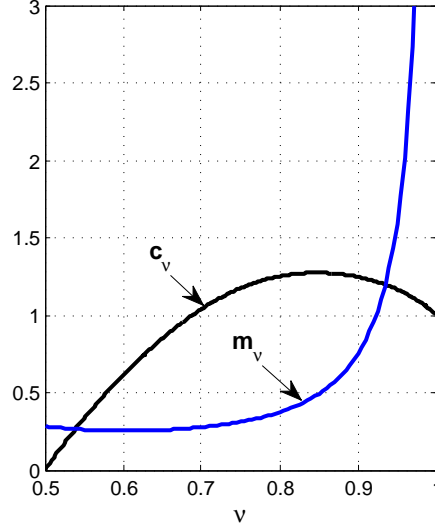


Figure 6: Maximum locations and maximum values of the Green function $\mathcal{G}_c(x; \nu)$: Plots of c_ν and m_ν for $1/2 \leq \nu \leq 1$

formula (21) was used. As expected, m_ν tends to infinity as ν tends to 1 that corresponds to the case of the wave equation.

Another interesting feature of the curve $m_\nu = m_\mu(\nu)$ that can be seen in Fig. 6 is that m_ν is first monotonically decreasing and then starts to increase. The minimum location of $m_\nu = m_\mu(\nu)$ is at $\nu \approx 0.61$ and the minimum value is nearly equal to 0.25. Whereas m_ν changes very slow on the interval $0 \leq \nu < 0.95$, it starts to rapidly grow in a small neighborhood of the point $\nu = 1$. It should be noted that despite of the fact that the curves $m_\nu = m_\nu(\nu)$ and $c_\nu = c_\nu(\nu)$ are not monotone and possess a minimum and a maximum, respectively, the product $c_\nu m_\nu$ is a monotone increasing function for all ν , $1/2 \leq \nu \leq 1$ (see Fig. 5).

Finally we note that most of the analytical results and numerical algorithms presented in this paper can be extended and applied for more general partial differential equations of fractional order in two and three dimensions. Another important point that should be investigated in future would be to give a physical and/or probabilistic interpretation of the results regarding the maximum location and maximum value of the Green function \mathcal{G}_c of the Cauchy problem for the time-fractional diffusion-wave equation. Especially,

an appropriate interpretation of the maximum of c_ν that is greater than 1 ($1 = c_1$) and the minimum of m_ν that is smaller than $m_{1/2} \approx 0.28$ would be desirable. The fact that the product of the maximum location $x_*(t)$ and the maximum value $\mathcal{G}_c^*(t; \nu)$ of the Green function $\mathcal{G}_c(x, t; \nu)$ is equal to the constant $c_\nu m_\nu$ for all $t > 0$ should be properly interpreted, too. All these matters will be considered elsewhere.

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